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Solving symmetric matrix word equations via symmetric space machinery

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Abstract

It has recently been proved by Hillar and Johnson [C.J. Hillar, C.R. Johnson, Symmetric word equations in two positive definite letters, Proc. Amer. Math. Soc. 132 (2004) 945–953] that every symmetric word equation in positive definite matrices has a positive definite solution (existence theorem). In this paper we partially solve the uniqueness conjecture via the symmetric space and non-positive curvature machinery existing in the open convex cone of positive definite matrices. Unique positive definite solutions are obtained in terms of geometric and weighted means and as fixed points of explicit strict contractions on the cone of positive definite matrices.

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1. Introduction

For a non-empty alphabet $\mathbb{A}$, we consider the concatenation monoid (semigroup with identity) of (*generalized*) *words* of the form

$$W = A_1^{p_1} A_2^{p_2} \cdots A_k^{p_k},$$

where each $A_j \in \mathbb{A}$ and each exponent p_j is a real number, subject to the standard exponential laws for adjacent powers with a common base. In particular $A_j^0 = I$, the identity, for each j. The

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reversal W^* of the word W is the word written in reverse order, and the word is *symmetric* (or "palindromic") if it is equal to its reversal.

A *symmetric word equation* for $\mathbb{A} = \{X, A, B\}$ (respectively, $\mathbb{A} = \{X, A_1, \dots, A_n, B\}$) is an equation of the form $W(X, A) = B$ (respectively, $W(X, A_1, \dots, A_n) = B$), where $W(X, A)$ is a symmetric word in X and A (respectively, $W(X, A_1, \dots, A_n)$ is a symmetric word in $X, A_1, \dots, A_n$); we further assume that the exponents of X are all positive, and other exponents are non-negative. We frequently write symmetric word equations with the notation $S(X, A) = B$, to suggest symmetry. Symmetric word equations arise naturally in matrix theory as equations over the cone of positive definite matrices. Such equations have recently been the topic of active investigation because of their relationship to the Bessis–Moussa–Villani trace conjecture, an open conjecture arising from statistical physics [10].

The simplest and best-known symmetric matrix word equation is the Riccati matrix equation $XAX = B$ which has a unique positive definite solution, the *geometric mean* $A^{-1}\#B$ of A^{-1} and B : $A^{-1}\#B = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$. This Riccati equation and the corresponding geometric mean appear in matrix inequalities (the harmonic–geometric–arithmetic mean inequalities, [1–3,5]), in semidefinite programming (scaling points, [9,20]) and in the geometry of symmetric spaces of non-compact type (geodesic midpoint, [12,13,15–17]).

A symmetric word equation $S(X, A) = B$ is called (*uniquely*) *solvable* if there exists (uniquely) a positive definite solution X of $S(X, A) = B$ for every pair of $n \times n$ positive definite matrices A and B. In [11], Hillar and Johnson proved that every positive definite symmetric word equation is solvable. They left open the uniqueness (of solution) problem.

In this paper we are interested in solving the uniqueness conjecture for a variety of symmetric word equations $S(X, A) = B$. Additionally we seek continuity of solutions as a function of the variables A and B over positive definite matrices. Our major contributions are threefold: first of all we demonstrate how the geometric mean and its generalization to weighted means can be used to give explicit solutions to certain classes of equations. Secondly we show how the geometry of the positive definite matrices equipped with a symmetric structure and a convex Riemannian metric allows the application of the Banach fixed point theorem to deduce solutions to other classes of symmetric equations. (We are unaware of this second technique being used previously to solve such equations.) Finally we show that equations through degree 5 are uniquely and continuously solvable. This is the best possible degree result, since it is shown in [6] that there are degree 6 equations that have multiple solutions.

As just asserted, the unique positive definite solutions we obtain here will often be realized as fixed points of strict contractions for the Riemannian distance metric on the cone of positive definite matrices. More specifically, the proof heavily depends on the non-positive curvature property of the Riemannian symmetric space of positive definite matrices, which is metric equivalent to the contraction property of the square root function: $\delta(A^{1/2}, B^{1/2}) \leqslant \frac{1}{2}\delta(A, B)$.

2. Invariant metrics on positive definite cones

Let Herm $(n, \mathbb{C})$ be the space of $n \times n$ Hermitian matrices. We recall that A *is positive semidefinite*, denoted by $0 \leqslant A$, if $\langle x, Ax\rangle \geqslant 0$ for all $x \in \mathbb{C}^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{C}^n$. Similarly, A is *positive definite*, denoted $0 < A$, if it is positive semidefinite and invertible, or equivalently, $\langle x, Ax\rangle > 0$ for all non-zero x. The *Löwner order* $A \leqslant B$ is defined by $A \leqslant B$ if and only if $B - A \geqslant 0$.

The set $\Omega = \Omega(n)$ of $n \times n$ positive definite Hermitian matrices is an open convex cone of Herm$(n, \mathbb{C})$ and is a typical example of a symmetric space of non-compact type. The symmetric space structure is given by $s_A(B) = AB^{-1}A$. We briefly review the Riemannian structure of Ω. See [12,13,18] for details.

The trace inner product $\langle X, Y\rangle := \mathrm{tr}(XY)$ on $\mathrm{Herm}(n, \mathbb{C})$ which is identified with the tangent space of Ω at I, gives rise to a natural Riemannian metric on Ω: the inner product on the tangent space of Ω at $A > 0$ is given by $\langle X, Y\rangle_A := \mathrm{tr}(A^{-1}XA^{-1}Y)$, $X, Y \in \mathrm{Herm}(n, \mathbb{R})$. The Riemannian metric distance $\delta(A, B)$ is given explicitly by

$$\delta(A, B) = \left(\sum_{i=1}^{n} \log^2 \lambda_i(A^{-1}B)\right)^{1/2},$$

where the $\lambda_i(A^{-1}B)$ denote the eigenvalues of $A^{-1}B$, and by direct computation it is invariant under the matrix inversion and congruence transformations:

$$\delta(A^{-1}, B^{-1}) = \delta(A, B) = \delta(M^*AM, M^*BM), \quad M \in \mathrm{GL}(n, \mathbb{C}). \tag{2.1}$$

The unique (Riemannian) geodesic curve passing through A at $t = 0$ and B at $t = 1A$ is given explicitly by

$$A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

and its geodesic midpoint coincides with the geometric mean of A and B, $A\#B := A\#_{1/2}B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$. The non-positive curvature property of the Riemannian manifold $\Omega(n)$ is known to be equivalent to the following inequality:

$$\delta(A^t, B^t) \leqslant t\delta(A, B), \quad 0 \leqslant t \leqslant 1. \tag{2.2}$$

It has recently been discovered by Bhatia [7] that the non-positive curvature property (2.2) holds for metrics inherited from symmetric gauge functions: if Φ is a symmetric gauge function, for instance the Schatten p-norms $\|x\|_p = (\sum |x_i|^p)^{1/p}$, then it defines a natural unitary invariant norm on the complex matrices $\|A\|_\Phi = \Phi(s_1(A), \ldots, s_n(A))$, where $s_i(A)$ are the singular values of A in non-increasing order. The corresponding metric distance on the positive definite cone Ω is determined by $\delta_\Phi(A, B) = \|\log(A^{-1}B)\|_\Phi$. The metric δ_Φ satisfies the non-positive curvature property (2.2). We note that $\delta(A, B) = \|\log(A^{-1}B)\|_2$.

3. The geometric mean and contractions

Lemma 3.1 (Riccati Lemma). *The geometric mean $A\#B$ of positive definite matrices A and B is the unique positive definite solution of the Riccati-type equation*

$$XA^{-1}X = B.$$

Furthermore, the geometric mean operation has the following properties:

(i) $A\#B = B\#A$,
(ii) $(A\#B)^{-1} = A^{-1}\#B^{-1}$,
(iii) $M(A\#B)M^* = (MAM^*)\#(MBM^*)$ *for any non-singular matrix M.*

Proof. See [1,11,13]. □

The non-positive curvature property (2.2) of the Riemannian metric on Ω is equivalent to (by its invariance property (2.1))

$$\delta(A\#_t X, A\#_t Y) \leqslant t\delta(X, Y) \quad \forall X, Y, A \in \Omega, \ 0 \leqslant t \leqslant 1. \tag{3.3}$$

A stronger result remains true: the distance function between two geodesies is convex

$$\delta(A\#_t B, C\#_t D) \leqslant (1-t)\delta(A, C) + t\delta(B, D), \quad t \in [0, 1]. \tag{3.4}$$

From the uniqueness of the geodesic line (cf. [4,8]) we conclude that

$$A\#_t B = B\#_{1-t} A, \quad t \in \mathbb{R}. \tag{3.5}$$

Using this fact and the triangular inequality, we have that

$$\begin{aligned}\delta(A\#_t B, C\#_t D) &\leqslant \delta(A\#_t B, A\#_t D) + \delta(A\#_t D, C\#_t D)\\ &\leqslant \delta(A\#_t B, A\#_t D) + \delta(D\#_{1-t} A, D\#_{1-t} C)\\ &\leqslant t\delta(B, D) + (1-t)\delta(A, C).\end{aligned}$$

A mapping $f : \Omega \to \Omega$ is *non-expansive* (respectively, a *contraction*) for the Riemannian metric δ if $\delta(f(X), f(Y)) \leqslant \delta(X, Y)$ (respectively, $\delta(f(X), f(Y)) < \delta(X, Y)$) for all $X, Y \in \Omega$. If there exists $0 < \alpha < 1$ such that $\delta(f(X), f(Y)) \leqslant \alpha\delta(X, Y)$ for all $X, Y \in \Omega$ then f is called a *strict contraction*. By completeness of the Riemannian metric, every strict contraction has a unique (positive definite) fixed point. The *least contraction coefficient* of a non-expansive function $f : \Omega \to \Omega$ is defined by

$$L(f) := \sup_{\substack{A,B\in\Omega\\ A\neq B}} \frac{\delta(f(A), f(B))}{\delta(A, B)}.$$

Lemma 3.2. *Let $f, g : \Omega \to \Omega$ be non-expansive functions for the Riemannian distance. Then for $t \in [0, 1]$, the map $f\#_t g$ defined by $(f\#_t g)(X) = f(X)\#_t g(X)$ is non-expansive and*

$$L(f\#_t g) \leqslant (1-t)L(f) + tL(g). \tag{3.6}$$

In particular, $f\#_t g$ is a strict contraction (and hence has a unique positive definite fixed point) if one of the following is satisfied:

(i) $0 < t < 1$ *and either f or g is a strict contraction,*
(ii) $t = 0$ *and f is a strict contraction,*
(iii) $t = 1$ *and g is a strict contraction.*

Proof. It follows from the convexity of the distance function (3.4) that

$$\begin{aligned}\delta((f\#_t g)(X), (f\#_t g)Y)) &= \delta(f(X)\#_t g(X), f(Y)\#_t g(Y))\\ &\leqslant (1-t)\delta(f(X), f(Y)) + t\delta(g(X), g(Y))\\ &\leqslant [(1-t)L(f) + tL(g)]\delta(X, Y). \qquad \square\end{aligned}$$

Remark 3.3. We use the notation $f_1\#_{t_1} f_2\#_{t_2} \cdots \#_{t_{k-1}} f_k$ in the usual way:

$$(f_1\#_{t_1} f_2\#_{t_2} \cdots \#_{t_{k-1}} f_k)(X) = f_1(X)\#_{t_1}(f_2\#_{t_2} \cdots \#_{t_{k-1}} f_k)(X)$$

although the geometric mean operation is not associative. If each f_i is non-expansive and $t_i \in [0, 1]$ then by (3.6) $f_1\#_{t_1} f_2\#_{t_2} \cdots \#_{t_{k-1}} f_k$ is non-expansive and

$$L(f_1\#_{t_1} f_2\#_{t_2} \cdots \#_{t_{k-1}} f_k) \leqslant \sum_{i-1}^{k} \left(\prod_{j=0}^{i-1} t_j\right)(1-t_i)L(f_i), \tag{3.7}$$

where $t_0 = 1$, $t_k = 0$. If one of f_i's is a strict contraction then $f_1 \#_{t_1} f_2 \#_{t_2} \cdots \#_{t_{k-1}} f_k$ is a strict contraction, provided each $t_i \in (0, 1)$ for $i = 1, \dots, k-1$.

For a positive definite matrix A, we denote by $C_A(X) = A$ the constant function, by $h_A(X) = AXA$ the congruence transformation and by $\tilde{h}_A(X)$ the function $X \mapsto (AXA)^{-1}$. Then C_A is a strict contraction and h_A and $\tilde{h}_A$ are isometries on Ω. We are interested in their weighted #-product: for example,

$$(h_A \#_{t_1} C_B \#_{t_2} \tilde{h}_C)(X) = (AXA) \#_{t_1} (B \#_{t_2} (CXC)^{-1}), \quad t_i \in (0, 1),$$

where A, B and C vary over positive definite matrices.

The following result appears in Proposition II.6 of [19].

Proposition 3.4. *Let (X, δ) be a complete metric space, $0 \leqslant \lambda < 1$, and*

$$C_\lambda = \{f : X \to X : L(f) \leqslant \lambda\}.$$

For $f \in C_\lambda$ let $p(f) \in X$ denote the unique fixed point of f. If we endow C_λ with the topology of pointwise convergence, then the fixed point map $p : C_\lambda \to X$ is continuous.

Proposition 3.5. *The composition $f \circ g$ of a finite weighted #-product f of $C_{A_i}, h_{A_i}, \tilde{h}_{A_i}, A_i > 0, 1 \leqslant i \leqslant m$ with at least one constant contraction factor C_{A_i} and a contraction g is a strict contraction. The unique positive definite fixed point depends continuously on its determining variables.*

Proof. Let $f = f_1 \#_{t_1} f_2 \#_{t_2} \cdots \#_{t_{k-1}} f_k$ where $t_i \in (0, 1)$, $i = 1, 2, \dots, k-1$ and each f_i is one of the form, C_{A_i}, h_{A_i} and $\tilde{h}_{A_i}$ and one of them is a constant contraction. Then from (3.7), $L(f) \leqslant \alpha < 1$ for some α, hence f is a strict contraction, and thus $f \circ g$ is a strict contraction.

Let A_{ij} be a sequence of positive definite matrices converging to A_i, and let f_{ij} be the map associated to A_{ij} of the same type as f_i, for instance, if $f_i = C_{A_i}$ then $f_{ij} = C_{A_{ij}}$. Since $C_{A_{ij}}, h_{A_{ij}}, \tilde{h}_{A_{ij}}$ converge (pointwise convergence) to $C_{A_i}, h_{A_i}, \tilde{h}_{A_i}$ respectively, since the weighted $\#_{t_i}$ operation is continuous, and since g is continuous, we have that

$$(f_{1j} \#_{t_1} f_{2j} \#_{t_2} \cdots \#_{t_{k-1}} f_{kj}) \circ g \to f \circ g = (f_1 \#_{t_1} f_2 \#_{t_2} \cdots \#_{t_{k-1}} f_k) \circ g$$

under the topology of pointwise convergence. Note that $L(f_{lj}) = L(f_l)$ for all $l = 1, 2, \dots, k$. Thus $L(f_{1j} \#_{t_1} f_{2j} \#_{t_2} \cdots \#_{t_{k-1}} f_{kj}) \leqslant \alpha < 1$ for all $j = 1, 2, \dots, \infty$, and hence its composition with g is also a strict contraction. We then apply Proposition 3.4 to conclude that the fixed point of the strict contraction $(f_{1j} \#_{t_1} f_{2j} \#_{t_2} \cdots \#_{t_{k-1}} f_{kj}) \circ g$ converges to the fixed point of $f \circ g = (f_1 \#_{t_1} f_2 \#_{t_2} \cdots \#_{t_{k-1}} f_k) \circ g$. $\square$

4. Unique solvability

Throughout this section we assume that *all matrices are $n \times n$ positive definite matrices and are measured by the Riemannian metric distance $\delta(A, B)$.*

A symmetric word equation $S(X, A) = B$ is called *uniquely and continuously solvable* if it is uniquely solvable and solutions depend continuously on (A, B). For instance, the Riccati equation $XAX = B$ is uniquely and continuously solvable from the continuity of the geometric mean operation

$$(A, B) \mapsto A^{-1} \# B = A^{-1/2} (A^{1/2} B A^{1/2})^{1/2} A^{-1/2}.$$

Two symmetric word equations are *uniquely equivalent* if for each, its unique solvability implies the unique solvability of the other. As observed in [11], $S(X^r, A^s) = B$ $(r > 0, s \neq 0)$, $S(X, A)^k = B$ (k a positive integer) and $A^s S(X, A) A^s = B$ are uniquely equivalent to $S(X, A) = B$, respectively.

We first solve the uniqueness conjecture for the simplest type of word equation

$$XAXA \cdots XAX = (XA)^m X = B. \tag{4.8}$$

We are indebted to Christopher Hillar for suggesting the formula for the solution. More generally, we have the following.

Theorem 4.1. *For any real number $t \neq -1$, the equation*

$$X(XAX)^t X = B$$

has the unique positive definite solution $X = \left(A^{-1}\#_{\frac{1}{1+t}} B\right)^{1/2}$. In particular for a positive integer m, the symmetric word equation (4.8)

$$(XA)^m X = B$$

has the unique positive definite solution $X = A^{-1}\#_{\frac{1}{m+1}} B$ and hence it is uniquely and continuously solvable.

Proof. We note from (3.5) that

$$X(XAX)^t X = X(X^{-1}A^{-1}X^{-1})^{-t} X = X^2 \#_{-t} A^{-1} = A^{-1}\#_{1+t} X^2.$$

Thus $X(XAX)^t X = B$ if and only if $B = A^{-1}\#_{1+t} X^2 = A^{-1/2}(A^{1/2}X^2A^{1/2})^{1+t}A^{-1/2}$ if and only if

$$X^2 = A^{-1/2}(A^{1/2}BA^{1/2})^{\frac{1}{1+t}}A^{-1/2} = A^{-1}\#_{\frac{1}{1+t}} B,$$

or

$$X = \left(A^{-1}\#_{\frac{1}{1+t}} B\right)^{1/2}.$$

The second assertion follows by observing

$$(XA)^m X = X^{1/2}(X^{1/2}AX^{1/2})^m X^{1/2}$$

and by the continuity of the geometric mean operation. □

Remark 4.2. It is interesting to note that the solution $A^{-1}\#_{\frac{1}{m}} B$ of symmetric word equation $(XA)^m X = B$ converges to A^{-1} as the degree m increases to infinity. On the other hand, the matrix B can be regarded as the limit of the sequence $A^{-1}\#_{\frac{2^m}{2^m+1}} B$, which is the unique positive definite solution of the equation

$$\underbrace{X\#(X\# \cdots \#(X}_{m}\#(XAX)) \cdots) = B.$$

Indeed by the preceding theorem $A^{-1}\#_{\frac{2^m}{2^m+1}} B$ is the unique solution of

$$\begin{aligned} B &= X^{1/2}(X^{1/2}AX^{1/2})^{\frac{1}{2^m}} X^{1/2} \\ &= X^{1/2}(\underbrace{I\#(I\# \cdots \#(I}_{m}\#(X^{-1/2}XAXX^{-1/2})) \cdots))X^{1/2} \\ &= \underbrace{X\#(X\# \cdots \#(X}_{m}\#(XAX)) \cdots), \end{aligned}$$

where the last equality follows from the homogeneity property of the geometric mean (Lemma 3.1).

Theorem 4.3. *Let s, t be real numbers with $t \neq 0$. The symmetric equation*

$$X^t A X^s A X^t = B \tag{4.9}$$

is uniquely and continuously solvable if either

(i) $|t| \geqslant |s|$ *or*
(ii) $1 \leqslant s/t \leqslant 3$.

Proof. Suppose that $|t| \geqslant |s|$. The case $s = 0$ is obvious from the Riccati Lemma, so we assume that $s \neq 0$. Set $Y = X^s$ so that (4.9) is uniquely equivalent to $Y^{t/s} AYAY^{t/s} = B$. By the Riccati Lemma, $Y^{t/s} = B\#(AYA)^{-1}$ and therefore $Y = (B\#(AYA)^{-1})^{s/t}$. Since $|s/t| \leqslant 1$, since inversion is an isometry, and by (2.2) and (3.3) the mapping $Y \mapsto (B\#(AYA)^{-1})^{s/t}$ is a strict contraction and has a unique fixed point depending continuously on (A, B) (by Proposition 3.5). Therefore the symmetric equation $Y^{t/s} AYAY^{t/s} = B$ and hence (4.9) has a unique positive definite solution depending continuously on (A, B).

Next, suppose that $1 \leqslant s/t \leqslant 3$. From

$$X^t A X^s A X^t = (X^t A X^t) X^{s-2t} (X^t A X^t)$$

and setting $Y = X^t$, (4.9) is uniquely equivalent to

$$(YAY) Y^{s/t-2} (YAY) = B. \tag{4.10}$$

By the Riccati Lemma, a solution of (4.10) corresponds to a fixed point of the mapping

$$Y \mapsto A^{-1}\#(B\#Y^{2-s/t}).$$

By Proposition 3.5 and by the condition on s and t, and (2.2), Eq. (4.9) has a unique positive definite solution depending continuously on (A, B). $\square$

Example 4.4. We consider two symmetric word equations

$$XAXAX^2AX^2AXAX = B \tag{4.11}$$

and

$$XAX^2AXAX^2AX = B. \tag{4.12}$$

Setting $Y = A^{1/2} X A^{1/2}$, we have

$$XAXAX^2AX^2AXAX = [(XA)^2X](XAX)[(XA)^2X] = A^{-1/2}Y^3A^{-1}Y^2A^{-1}Y^3A^{-1/2}$$

and

$$XAX^2AXAX^2AX = (XAX)[(XA)^2X](XAX) = A^{-1/2}Y^2A^{-1}Y^3A^{-1}Y^2A^{-1/2}.$$

Thus (4.11) and (4.12) are uniquely equivalent to $X^3AX^2AX^3 = B$ and $X^2AX^3AX^2 = B$, respectively. Applying the preceding theorem with $(t = 3, s = 2)$ and $(s = 3, t = 2)$, we see that the symmetric word equations (4.11) and (4.12) are uniquely and continuously solvable.

Theorem 4.5. *For positive definite matrices A and B, the matrix equation*

$$(XAX)^t X^s (XAX)^t = B \tag{4.13}$$

has a unique positive definite solution depending continuously on (A, B) *if* $|s| \leqslant 1$ *and* $|t| \geqslant 1$.

Proof. It follows from the Riccati Lemma that $X > 0$ is a solution of (4.13) if and only if X is a positive definite fixed point of the map $X \mapsto A^{-1}\#(B\#X^{-s})^{1/t}$. By Proposition 3.5 and by the condition on s and t, (4.13) has a unique positive definite solution depending continuously on (A, B). □

Applying the preceding theorem with $s = t = 1$ we have

Example 4.6. The symmetric word equation $XAX^3AX = B$ is uniquely and continuously solvable.

In the following, we consider symmetric word equations $S(X, A)$ with (positive) integer exponents of X. Up to unique equivalence it must then be of the form

$$\underbrace{X^{p_1}A^{q_1}X^{p_2}A^{q_2}\cdots X^{p_k}A^{q_k}}_{=W} X^{p_{k+1}} \underbrace{A^{q_k}X^{p_k}\cdots A^{q_2}X^{p_2}A^{q_1}X^{p_1}}_{=W^*} = B, \tag{4.14}$$

where the p_i's are all positive integers but p_{k+1} could be zero. The sum of all exponents of X is called the *degree* of the symmetric word equation (4.14) and denoted by $d(S(X, A))$:

$$d(S(X, A)) := p_{k+1} + \sum_{i=1}^{k} 2p_i.$$

Theorem 4.7. *Every symmetric word equation of degree* $\leqslant 5$ *is uniquely and continuously solvable.*

Proof. We first observe that the word equations

$$XAX^kAX = B, \quad X^2(AX)^{k-2}AX^2 = B \tag{4.15}$$

are uniquely equivalent via the substitution in the second equation $Y = A^{1/2}XA^{1/2}$, minor rearrangement, and replacing A^{-1} by A [11].

We give a proof by considering all possible (up to equivalence) symmetric word equations of the given degree. We exclude the trivial equations of the form $X^n = B$.

(1) $d(S(X, A)) = 2$. In this case, $XAX = B$ is the only symmetric word equation. By the Riccati Lemma,

$$XAX = B \Leftrightarrow X = A^{-1}\#B.$$

(2) $d(S(X, A)) = 3$. The only non-trivial symmetric word equation is of the form

$$XAXAX = B$$

which has a unique positive definite solution $X = A^{-1}\#_{1/3}B$ by Theorem 4.1.

(3) $d(S(X, A)) = 4$. We have two non-equivalent symmetric word equations of degree 4:

$$XAXA^qXAX = B, \tag{4.16}$$

$$XAX^2AX = B. \tag{4.17}$$

We note that the word equation $X^2AX^2 = B$ is uniquely equivalent to $XAX = B$. Both Eqs. (4.16) and (4.17) have explicit unique solution by the Riccati Lemma: Eq. (4.16) has solution

$$XAX = B\#A^{-q} \Leftrightarrow X = A^{-1}\#(A^{-q}\#B)$$

and the second equation (4.17) has solution

$$(XAX)^2 = B \Leftrightarrow X = A^{-1}\#B^{1/2}.$$

(4) $d(S(X, A)) = 5$. There are two non-equivalent symmetric word equations of degree 5:

$$XAXA^qXA^qXAX = B, \tag{4.18}$$

$$XAX^3AX = B. \tag{4.19}$$

We note that $X^2AXAX^2 = B$ is equivalent to (4.19) from (4.15).

By Example 4.6, the symmetric equation (4.19) is uniquely and continuously solvable. Eq. (4.18) is equivalent by the Riccati Lemma to the fixed point problem

$$XAX = (A^qXA^q)^{-1}\#B \Leftrightarrow X = A^{-1}\#((A^qXA^q)^{-1}\#B).$$

By Proposition 3.5, the map $X \mapsto A^{-1}\#((A^qXA^q)^{-1}\#B)$ has a unique fixed point depending continuously on (A, B). $\square$

Remark 4.8. The preceding theorem easily generalizes to "interlaced" symmetric equations (symmetric equations with multiple constants where two distinct constants do not appear side-by-side) of the form $S(X, A_1, \ldots, A_n) = B$ of degree less than or equal to 5. The only changes that take place in the preceding equations is that Eq. (4.15) is replaced by

$$XA_1XA_2XA_1X = B$$

and Eq. (4.17) is replaced by

$$XA_1XA_2XA_2XA_1X = B.$$

The same proof techniques remain valid in this more general setting.

Remark 4.9. The preceding theorem is in some sense the best possible, since it is shown in [6] that there exist equations of degree 6 with multiple solutions. Indeed the equation

$$S(X, A) = XAX^2A^3X^2AX = B$$

has multiple solutions for a suitable choice of A and B.

5. Concluding remarks

The positive results derived in this paper readily extend to more general contexts. In particular, the set of positive definite elements of the form xx^*, x invertible, in a C^*-algebra forms a symmetric cone with seminegative metric curvature ($\delta(A^{1/2}, B^{1/2}) \leqslant \frac{1}{2}\delta(A, B)$) with respect to the Thompson or part metric δ (see, e.g., [14]). Thus all the arguments remain valid in this setting.

More generally, one can work with the symmetric space with seminegative metric curvature as considered in [14]. In this case one replaces the symmetric words by iterates of the quadratic representation operator, i.e., $XAXAX$ is replaced by $P_X(P_A(X))$, where $P_X = S_X S_I$, S_X and S_I are the point reflection through X and I respectively, and I is a distinguished point (chosen to be the identity in the algebra case). See [14] for further details.

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